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# Classifying the Clique-Width of $H$ -Free Bipartite Graphs<sup>\*</sup>

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**Abstract.** Let  $G$  be a bipartite graph, and let  $H$  be a bipartite graph with a fixed bipartition  $(B_H, W_H)$ . We consider three different, natural ways of forbidding  $H$  as an induced subgraph in  $G$ . First,  $G$  is  $H$ -free if it does not contain  $H$  as an induced subgraph. Second,  $G$  is strongly  $H$ -free if  $G$  is  $H$ -free or else has no bipartition  $(B_G, W_G)$  with  $B_H \subseteq B_G$  and  $W_H \subseteq W_G$ . Third,  $G$  is weakly  $H$ -free if  $G$  is  $H$ -free or else has at least one bipartition  $(B_G, W_G)$  with  $B_H \not\subseteq B_G$  or  $W_H \not\subseteq W_G$ . Lozin and Volz characterized all bipartite graphs  $H$  for which the class of strongly  $H$ -free bipartite graphs has bounded clique-width. We extend their result by giving complete classifications for the other two variants of  $H$ -freeness.

## 1 Introduction

The *clique-width* of a graph  $G$ , is a well-known graph parameter that has been studied both in a structural and in an algorithmic context. It is the minimum number of labels needed to construct  $G$  by using the following four operations:

- (i) creating a new graph consisting of a single vertex  $v$  with label  $i$ ;
- (ii) taking the disjoint union of two labelled graphs  $G_1$  and  $G_2$ ;
- (iii) joining each vertex with label  $i$  to each vertex with label  $j$  ( $i \neq j$ );
- (iv) renaming label  $i$  to  $j$ .

We refer to the surveys of Gurski [13] and Kamiński, Lozin and Milanič [14] for an in-depth study of the properties of clique-width.

We say that a class of graphs has *bounded* clique-width if every graph from the class has clique-width at most  $p$  for some constant  $p$ . As many NP-hard graph problems can be solved in polynomial time on graph classes of bounded clique-width [10,15,20,21], it is natural to determine whether a certain graph class has bounded clique-width and to find new graph classes of bounded clique-width. In particular, many papers determined the clique-width of graph classes characterized by one or more forbidden induced subgraphs [1,2,3,4,5,6,7,8,9,11,16,17,18,19].

In this paper we focus on classes of bipartite graphs characterized by a forbidden induced subgraph  $H$ . A graph  $G$  is  $H$ -free if it does not contain  $H$  as

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an induced subgraph. If  $G$  is bipartite, then when considering notions for  $H$ -freeness, we may assume without loss of generality that  $H$  is bipartite as well. For bipartite graphs, the situation is more subtle as one can define the notion of freeness with respect to a fixed bipartition  $(B_H, W_H)$  of  $H$ . This leads to two other notions (see also Section 2 for formal definitions). We say that a bipartite graph  $G$  is strongly  $H$ -free if  $G$  is  $H$ -free or else has no bipartition  $(B_G, W_G)$  with  $B_H \subseteq B_G$  and  $W_H \subseteq W_G$ . Strongly  $H$ -free graphs have been studied with respect to their clique-width, although under less explicit terminology (see e.g. [14,17,18]). In particular, Lozin and Volz [18] completely determined those bipartite graphs  $H$ , for which the class of strongly  $H$ -free graphs has bounded clique-width (we give an exact statement of their result in Section 3). If  $G$  is  $H$ -free or else has at least one bipartition  $(B_G, W_G)$  with  $B_H \not\subseteq B_G$  or  $W_H \not\subseteq W_G$ , then  $G$  is said to be weakly  $H$ -free. As far as we are aware this notion has not been studied with respect to the clique-width of bipartite graphs.

**Our Results:** We completely classify the classes of  $H$ -free bipartite graphs of bounded clique-width. We also introduce the notion of weak  $H$ -freeness for bipartite graphs and characterize those classes of weakly  $H$ -free bipartite graphs that have bounded clique-width. In this way, we have identified a number of new graph classes of bounded clique-width. Before stating our results precisely in Section 3, we first give some terminology and examples in Section 2. In Section 4 we give the proofs of our results.

## 2 Terminology and Examples

We first give some terminology on general graphs, followed by terminology for bipartite graphs. We illustrate the definitions of  $H$ -freeness, strong  $H$ -freeness and weak  $H$ -freeness of bipartite graphs with some examples. As we will explain, these examples also make clear that all three notions are different from each other.

**General graphs:** Let  $G$  and  $H$  be graphs. We write  $H \subseteq_i G$  to indicate that  $H$  is an induced subgraph of  $G$ . A bijection of the vertices  $f : V_G \rightarrow V_H$  is called a (*graph*) *isomorphism* when  $uv \in E_G$  if and only if  $f(u)f(v) \in E_H$ . If such a bijection exists then  $G$  and  $H$  are *isomorphic*. Let  $\{H_1, \dots, H_p\}$  be a set of graphs. A graph  $G$  is  $(H_1, \dots, H_p)$ -free if no  $H_i$  is an induced subgraph of  $G$ . If  $p = 1$  we may write  $H_1$ -free instead of  $(H_1)$ -free. The *disjoint union*  $G + H$  of two vertex-disjoint graphs  $G$  and  $H$  is the graph with vertex set  $V_G \cup V_H$  and edge set  $E_G \cup E_H$ . We denote the disjoint union of  $r$  copies of  $G$  by  $rG$ .

**Bipartite graphs:** A graph  $G$  is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. Let  $H$  be a bipartite graph. We say that  $H$  is a *labelled* bipartite graph if we are also given a *black-and-white labelling*  $\ell$ , which is a labelling that assigns either the colour “black” or the colour “white” to each vertex of  $H$  in such a way that the two resulting monochromatic colour classes  $B_H^\ell$  and  $W_H^\ell$  form a partition of  $H$  into two (possibly empty) independent sets. From now on we denote a graph  $H$  with such a labelling  $\ell$  by

$H^\ell = (B_H^\ell, W_H^\ell, E_H)$ . Here the pair  $(B_H^\ell, W_H^\ell)$  is *ordered*, that is,  $(B_H^\ell, W_H^\ell, E_H)$  and  $(W_H^\ell, B_H^\ell, E_H)$  are different labelled bipartite graphs.

We say that two labelled bipartite graphs  $H_1^\ell$  and  $H_2^{\ell^*}$  are *isomorphic* if the (unlabelled) graphs  $H_1$  and  $H_2$  are isomorphic, and if in addition there exists an isomorphism  $f : V_{H_1} \rightarrow V_{H_2}$  such that for all  $u \in V_{H_1}$ ,  $u \in W_{H_1}^\ell$  if and only if  $f(u) \in W_{H_2}^{\ell^*}$ . Moreover, if  $H_1 = H_2$ , then  $\ell$  and  $\ell^*$  are said to be *isomorphic* labellings. For example, the bipartite graphs  $(\{u, v\}, \emptyset)$  and  $(\{x, y\}, \emptyset)$  are isomorphic, and the labelled bipartite graph  $(\{u, v\}, \emptyset, \emptyset)$  is isomorphic to the labelled bipartite graph  $(\{x, y\}, \emptyset, \emptyset)$ . However,  $(\{x, y\}, \emptyset, \emptyset)$  is neither isomorphic to  $(\emptyset, \{x, y\}, \emptyset)$  nor to  $(\{x\}, \{y\}, \emptyset)$  (also see Fig. 1).

We write  $H_1^\ell \subseteq_{li} H_2^{\ell^*}$  if  $H_1 \subseteq_i H_2$ ,  $B_{H_1}^\ell \subseteq B_{H_2}^{\ell^*}$  and  $W_{H_1}^\ell \subseteq W_{H_2}^{\ell^*}$ . In this case we say that  $H_1^\ell$  is a *labelled induced subgraph* of  $H_2^{\ell^*}$ . Note that the two labelled bipartite graphs  $H_1^{\ell_1}$  and  $H_2^{\ell_2}$  are isomorphic if and only if  $H_1^{\ell_1}$  is a labelled induced subgraph of  $H_2^{\ell_2}$ , and vice versa.



Fig. 1: The graph  $2P_1$  partitioned into three ways; none of these three labelled bipartite graphs are isomorphic to each other.

Let  $G$  be an (unlabelled) bipartite graph, and let  $H^\ell$  be a labelled bipartite graph. We say that  $G$  contains  $H^\ell$  as a *strongly labelled induced subgraph* if  $H^\ell \subseteq_{li} (B_G, W_G, E_G)$  for some bipartition  $(B_G, W_G)$  of  $G$ . If not, then  $G$  is said to be *strongly  $H^\ell$ -free*. We say that  $G$  contains  $H^\ell$  as a *weakly labelled induced subgraph* if  $H^\ell \subseteq_{li} (B_G, W_G, E_G)$  for all bipartitions  $(B_G, W_G)$  of  $G$ . If not, then  $G$  is said to be *weakly  $H^\ell$ -free*. Equivalently,  $G$  is strongly  $H^\ell$ -free if for every labelling  $\ell^*$  of  $G$ ,  $G^{\ell^*}$  does not contain  $H^\ell$  as a labelled induced subgraph and  $G$  is weakly  $H^\ell$ -free if there is a labelling  $\ell^*$  of  $G$  such that  $G^{\ell^*}$  does not contain  $H^\ell$  as a labelled induced subgraph. Note that these two notions of freeness are only defined for (unlabelled) bipartite graphs. Let  $\{H_1^{\ell_1}, \dots, H_p^{\ell_p}\}$  be a set of labelled bipartite graphs. Then a graph  $G$  is *strongly (weakly)  $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$ -free* if  $G$  is strongly (weakly)  $H_i^{\ell_i}$ -free for  $i = 1, \dots, p$ .

The following lemma shows that for all labelled bipartite graphs  $H^\ell$ , the class of  $H$ -free graphs is a (possibly proper) subclass of the class of strongly  $H^\ell$ -free bipartite graphs and that the latter graph class is a (possibly proper) subclass of the class of weakly  $H^\ell$ -free bipartite graphs.

**Lemma 1.** *Let  $G$  be a bipartite graph and  $H^\ell$  be a labelled bipartite graph. The following two statements hold:*

- (i) *If  $G$  is  $H$ -free, then  $G$  is strongly  $H^\ell$ -free.*
- (ii) *If  $G$  is strongly  $H^\ell$ -free, then  $G$  is weakly  $H^\ell$ -free.*

*Moreover, the two reverse statements are not necessarily true.*

*Proof.* Statements (i) and (ii) follow by definition. The following two examples, which are also depicted in Fig. 2, show that the reverse statements may not necessarily be true. Let  $G$  be isomorphic to  $S_{1,1,3}$  with  $V_G = \{u_1, \dots, u_6\}$  and  $E_G = \{u_1u_2, u_1u_3, u_1u_4, u_4u_5, u_5u_6\}$ . Let  $H = K_{1,3} + P_1$ . We denote the vertex set and edge set of  $H$  by  $V_H = \{x_1, x_2, x_3, x_4, x_5\}$  and  $E_H = \{x_1x_2, x_1x_3, x_1x_4\}$ .

Let  $H^\ell = (\{x_2, x_3, x_4\}, \{x_1, x_5\}, E_H)$ . We first notice that  $G$  is not  $H$ -free, because  $G[u_1, u_2, u_3, u_4, u_6]$  is isomorphic to  $K_{1,3} + P_1$ . However, we do have that  $G$  is strongly  $H^\ell$ -free, because  $H^\ell$  is neither a labelled induced subgraph of  $(\{u_1, u_5\}, \{u_2, u_3, u_4, u_6\}, E_G)$  nor of  $(\{u_2, u_3, u_4, u_6\}, \{u_1, u_5\}, E_G)$ .

Let  $H^{\ell^*} = (\{x_2, x_3, x_4, x_5\}, \{x_1\}, E_H)$ . Then  $G$  is not strongly  $H^{\ell^*}$ -free, because  $(\{u_2, u_3, u_4, u_6\}, \{u_1\}, \{u_1u_2, u_1u_3, u_1u_4\})$  is isomorphic to  $H^{\ell^*}$ . However,  $G$  is weakly  $H^{\ell^*}$ -free, because  $H^{\ell^*}$  is not a labelled induced subgraph of  $(\{u_1, u_5\}, \{u_2, u_3, u_4, u_6\}, E_G)$ .  $\square$

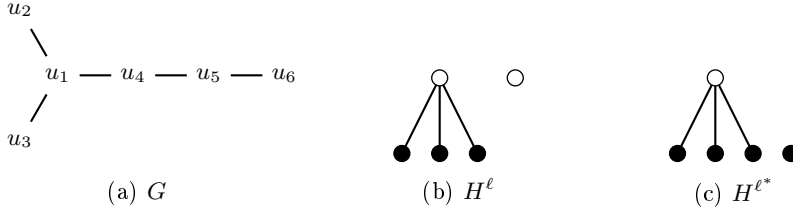


Fig. 2: The graphs  $G, H^\ell$  and  $H^{\ell^*}$  from the proof of Lemma 1.

**Special Graphs:** For  $r \geq 1$ , the graphs  $C_r, K_r, P_r$  denote the cycle, complete graph and path on  $r$  vertices, respectively, and the graph  $K_{1,r}$  denotes the star on  $r+1$  vertices. If  $r = 3$ , the graph  $K_{1,r}$  is also called the *claw*. For  $1 \leq h \leq i \leq j$ , let  $S_{h,i,j}$  denote the tree that has only one vertex  $x$  of degree 3 and that has exactly three leaves, which are of distance  $h, i$  and  $j$  from  $x$ , respectively. Observe that  $S_{1,1,1} = K_{1,3}$ . A graph  $S_{h,i,j}$  is called a *subdivided claw*.

Let  $H^\ell = (B_H^\ell, W_H^\ell, E_H)$  be a labelled bipartite graph. The *opposite* of  $H^\ell$  is defined as the labelled bipartite graph  $H^{\bar{\ell}} = (W_H^\ell, B_H^\ell, E_H)$ . We say that  $\bar{\ell}$  is the *opposite* black-and-white labelling of  $\ell$ . Suppose that  $H$  is a bipartite graph such that among all its black-and-white labellings, all those that maximize the number of black vertices are isomorphic. In this case we pick one of such labelling and call it  $b$ .

### 3 The Classifications

A full classification of the boundedness of the clique-width of strongly  $H^\ell$ -free bipartite graphs was given by Lozin and Voltz [18], except that in their result the trivial case when  $H^\ell = (sP_1)^b$  or  $H^\ell = (sP_1)^{\bar{b}}$  for some  $s \geq 1$  was missing. Their proof is correct except that it overlooked this case, which occurs when one of the colour classes of the labelled graph  $H^\ell$  is empty. However, strongly  $(sP_1)^b$ -free

bipartite graphs can have at most  $2s - 2$  vertices, and as such form a class of bounded clique-width. Below we state their result after incorporating this small correction, followed by our results for the other two variants of freeness. We refer to Fig. 3 for pictures of the labelled bipartite graphs used in Theorems 1 and 3.

**Theorem 1 ([18]).** *Let  $H^\ell$  be a labelled bipartite graph. The class of strongly  $H^\ell$ -free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H^\ell = (sP_1)^b$       or     $H^\ell = (sP_1)^{\bar{b}}$       for some  $s \geq 1$
- $H^\ell \subseteq_{li} (K_{1,3} + 3P_1)^b$     or     $H^\ell \subseteq_{li} (K_{1,3} + 3P_1)^{\bar{b}}$
- $H^\ell \subseteq_{li} (K_{1,3} + P_2)^b$     or     $H^\ell \subseteq_{li} (K_{1,3} + P_2)^{\bar{b}}$
- $H^\ell \subseteq_{li} (P_1 + S_{1,1,3})^b$     or     $H^\ell \subseteq_{li} (P_1 + S_{1,1,3})^{\bar{b}}$
- $H^\ell \subseteq_{li} (S_{1,2,3})^b$       or     $H^\ell \subseteq_{li} (S_{1,2,3})^{\bar{b}}$ .

**Theorem 2.** *Let  $H$  be a graph. The class of  $H$ -free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H = sP_1$  for some  $s \geq 1$
- $H \subseteq_i K_{1,3} + 3P_1$
- $H \subseteq_i K_{1,3} + P_2$
- $H \subseteq_i P_1 + S_{1,1,3}$
- $H \subseteq_i S_{1,2,3}$ .

**Theorem 3.** *Let  $H^\ell$  be a labelled bipartite graph. The class of weakly  $H^\ell$ -free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H^\ell = (sP_1)^b$       or     $H^\ell = (sP_1)^{\bar{b}}$       for some  $s \geq 1$
- $H^\ell \subseteq_{li} (P_1 + P_5)^b$     or     $H^\ell \subseteq_{li} (P_1 + P_5)^{\bar{b}}$
- $H \subseteq_i P_2 + P_4$
- $H \subseteq_i P_6$ .

## 4 The Proofs of Our Results

We first recall a number of basic facts on clique-width known from the literature. We then state a number of other lemmas which we use to prove Theorems 2 and 3.

### 4.1 Facts about Clique-width

The *bipartite complement* of a bipartite graph *with respect to* a bipartition  $(B, W)$  is the bipartite graph with bipartition  $(B, W)$ , in which two vertices  $u \in B$  and  $v \in W$  are adjacent if and only if  $uv \notin E$ . For instance, the graph  $2P_2$  has  $C_4$  as its only bipartite complement, whereas the graph  $2P_1$  has  $2P_1$  and  $P_2$  as its bipartite complements. For two disjoint vertex subsets  $X$  and  $Y$  in  $G$ , the *bipartite complementation* operation with respect to  $X$  and  $Y$  acts on  $G$

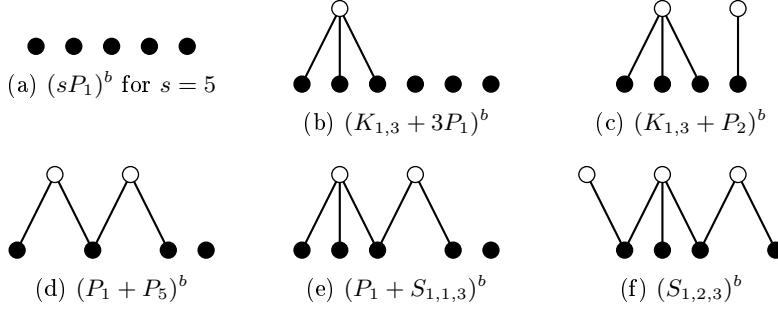


Fig. 3: The labelled bipartite graphs used in Theorems 1 and 3.

by replacing every edge with one end-vertex in  $X$  and the other one in  $Y$  by a non-edge and vice versa. The *edge subdivision* operation replaces an edge  $vw$  in a graph by a new vertex  $u$  with edges  $uv$  and  $uw$ .

We now state some useful facts for dealing with clique-width. We will use these facts throughout the paper. We will say that a graph operation *preserves* boundedness of clique-width if for every constant  $k$  and every graph class  $\mathcal{G}$ , the graph class  $\mathcal{G}_{[k]}$  obtained by performing the operation at most  $k$  times on each graph in  $\mathcal{G}$  has bounded clique-width if and only if  $\mathcal{G}$  has bounded clique-width.

*Fact 1.* Vertex deletion preserves boundedness of clique-width [16].

*Fact 2.* Bipartite complementation preserves boundedness of clique-width [14].

*Fact 3.* For a class of graphs  $\mathcal{G}$  of *bounded* degree, let  $\mathcal{G}'$  be the class of graphs obtained from  $\mathcal{G}$  by applying zero or more edge subdivision operations to each graph in  $\mathcal{G}$ . Then  $\mathcal{G}$  has bounded clique-width if and only if  $\mathcal{G}'$  has bounded clique-width [14].

We also use some other elementary results on the clique-width of graphs. In order to do so we need the notion of a *wall*. We do not formally define this notion, but instead refer to Fig. 4, in which three examples of walls of different height are depicted. A *k-subdivided wall* is a graph obtained from a wall after subdividing each edge exactly  $k$  times for some constant  $k \geq 0$ . The next well-known lemma follows from combining Fact 3 with the fact that walls have maximum degree 3 and unbounded clique-width (see e.g. [14]).

**Lemma 2.** *For every constant  $k$ , the class of  $k$ -subdivided walls has unbounded clique-width.*

We let  $\mathcal{S}$  be the class of graphs each connected component of which is either a subdivided claw  $S_{h,i,j}$  for some  $1 \leq h \leq i \leq j$  or a path  $P_r$  for some  $r \geq 1$ . This leads to the following lemma, which is well-known and follows from the fact that walls have maximum degree at most 3 and from Lemma 2 by choosing an appropriate value for  $k$  (also note that  $k$ -subdivided walls are bipartite for all  $k \geq 0$ ).

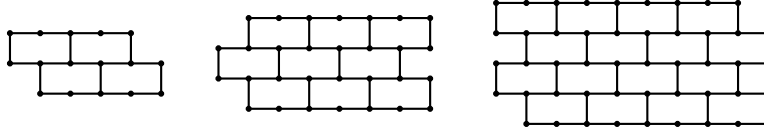


Fig. 4: Walls of height 2, 3, and 4, respectively.

**Lemma 3.** *Let  $\{H_1, \dots, H_p\}$  be a finite set of graphs. If  $H_i \notin \mathcal{S}$  for  $i = 1, \dots, p$  then the class of  $(H_1, \dots, H_p)$ -free bipartite graphs has unbounded clique-width.*

#### 4.2 A Number of Other Lemmas

We start with a lemma which is related to Lemma 1 and which follows immediately from the corresponding definitions.

**Lemma 4.** *Let  $G$  and  $H$  be bipartite graphs. Then  $G$  is  $H$ -free if and only if  $G$  is strongly  $H^\ell$ -free for all black-and-white labellings  $\ell$  of  $H$ .*

A graph  $G$  that contains a graph  $H$  as an induced subgraph may be weakly  $H^\ell$ -free for all black-and-white labellings  $\ell$  of  $H$ ; take for instance the graphs  $G$  and  $H$  from the proof of Lemma 1. However, we can make the following observation, which also follows directly from the corresponding definitions.

**Lemma 5.** *Let  $H$  be a bipartite graph with a unique black-and-white labelling  $\ell$  (up to isomorphism). Then every bipartite graph  $G$  is  $H$ -free if and only if it is weakly  $H^\ell$ -free.*

Note that there exist both connected bipartite graphs (for example  $H = P_6$ ) and disconnected bipartite graphs (for example  $H = 2P_2$ ) that satisfy the condition of Lemma 5.

Two black-and-white labellings of a bipartite graph  $H$  are said to be *equivalent* if they are isomorphic or opposite to each other; otherwise they are said to be *non-equivalent*. The following lemma follows directly from the definitions.

**Lemma 6.** *Let  $\ell$  and  $\ell^*$  be two equivalent black-and-white labellings of a bipartite graph  $H$ . Then the class of strongly (weakly)  $H^\ell$ -free graphs is equal to the class of strongly (weakly)  $H^{\ell^*}$ -free graphs.*

The following lemma is due to Lozin and Rautenbach [17].

**Lemma 7 ([17]).** *Let  $\{H_1^{\ell_1}, \dots, H_p^{\ell_p}\}$  be a finite set of labelled bipartite graphs. For  $i = 1, \dots, p$ , let  $F_i$  denote the bipartite complement of  $H_i$  with respect to  $(B_{H_i}^{\ell_i}, W_{H_i}^{\ell_i})$ . If  $H_i \notin \mathcal{S}$  for all  $1 \leq i \leq p$  or  $F_i \notin \mathcal{S}$  for all  $1 \leq i \leq p$ , then the class of strongly  $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$ -free bipartite graphs has unbounded clique-width.*



In the next lemma we demonstrate a list of  $H$ -free bipartite classes with unbounded clique-width. It is obtained by combining a known result of Lozin and Voltz [18] with a number of new results.

**Lemma 8.** *The class of  $H$ -free bipartite graphs has unbounded clique-width if  $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3\}$ .*

*Proof.* Lozin and Voltz [18] showed that  $2P_3$ -free bipartite graphs have unbounded clique-width. Let  $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2\}$ , and let  $\{H^{\ell_1}, \dots, H^{\ell_p}\}$  be the set of all non-equivalent labelled bipartite graphs isomorphic to  $H$ . For  $i = 1, \dots, p$ , let  $F_i$  denote the bipartite complement of  $H$  with respect to  $(B_H^{\ell_i}, W_H^{\ell_i})$ . We will show that every  $F_i$  does not belong to  $\mathcal{S}$ . Then, by Lemma 7 the class of strongly  $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$ -free bipartite graphs has unbounded clique-width. Because a bipartite graph is  $H$ -free if and only if it is strongly  $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$ -free (by Lemmas 4 and 6), this means that the class of  $H$ -free bipartite graphs has unbounded clique-width.

Suppose  $H \in \{2P_1 + 2P_2, 2P_1 + P_4\}$ . Let  $V_H = \{x_1, \dots, x_6\}$  with  $E_H = \{x_1x_2, x_3x_4\}$  if  $H = 2P_1 + 2P_2$  and  $E_H = \{x_1x_2, x_2x_3, x_3x_4\}$  if  $H = 2P_1 + P_4$ . Then  $H$  has only two non-equivalent black-and-white labellings. We may assume without loss of generality that one of these two labellings colours  $x_1, x_3, x_5, x_6$  black and  $x_2, x_4$  white, whereas the other one colours  $x_1, x_3, x_5$  black and  $x_2, x_4, x_6$  white. Let  $F_1$  and  $F_2$  be the bipartite complements corresponding to the first and second labellings, respectively. The vertices  $x_2, x_4, x_5, x_6$  induce a  $C_4$  in  $F_1$ , whereas the vertices  $x_1, x_4, x_5, x_6$  induce a  $C_4$  in  $F_2$ . Hence,  $F_1$  and  $F_2$  do not belong to  $\mathcal{S}$ .

Suppose  $H = 4P_1 + P_2$ . Let  $V_H = \{x_1, \dots, x_6\}$  and  $E_H = \{x_1x_2\}$ . Then  $H$  has three non-equivalent black-and-white labellings. We may assume without loss of generality that the first one colours  $x_1, x_3, x_4, x_5, x_6$  black and  $x_2$  white, the second one colours  $x_1, x_3, x_4, x_5$  black and  $x_2, x_6$  white, and the third one colours  $x_1, x_3, x_4$  black and  $x_2, x_5, x_6$  white. Let  $F_1, F_2, F_3$  denote the corresponding bipartite complements. The vertices  $x_2, \dots, x_6$  induce a  $K_{1,4}$  in  $F_1$ . The vertices  $x_2, x_3, x_4, x_6$  induce a  $C_4$  in  $F_2$  and  $F_3$ . Hence, none of  $F_1, F_2, F_3$  belongs to  $\mathcal{S}$ .

Suppose  $H = 3P_2$ . Let  $V_H = \{x_1, \dots, x_6\}$  and  $E_H = \{x_1x_2, x_3x_4, x_5x_6\}$ . Let  $\ell$  be a black-and-white labelling of  $H$  that colours  $x_1, x_3, x_5$  black and  $x_2, x_4, x_6$  white. Then every other labelling  $\ell^*$  of  $H$  is isomorphic to  $\ell$ . The bipartite complement of  $H$  with respect to  $(B_H^\ell, W_H^\ell)$  is isomorphic to  $C_6$ , which does not belong to  $\mathcal{S}$ .  $\square$

We will also need the following lemma. We omit the proof due to space restrictions.

**Lemma 9.** *Let  $H \in \mathcal{S}$ . Then  $H$  is  $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$ -free if and only if  $H = sP_1$  for some integer  $s \geq 1$  or  $H$  is an induced subgraph of one of the graphs in  $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$ .*

The last lemma we need before proving the main results of this paper is the following one (we use it several times in the proof of Theorem 3).

**Lemma 10.** *Let  $H^\ell$  be a labelled bipartite graph. The class of weakly  $H^\ell$ -free bipartite graphs has unbounded clique-width in both of the following cases:*

- (i)  $H^\ell$  contains a vertex of degree at least 3, or
- (ii)  $H^\ell$  contains four independent vertices, not all of the same colour.

*Proof.* Let  $b_1$  be a black-and-white labelling of  $4P_1$  that colours three vertices black and one vertex white. Let  $b_2$  be a black-and-white labelling of  $4P_1$  that colours two vertices black and two vertices white. We show below that the class of weakly  $H^\ell$ -free bipartite graphs has unbounded clique-width if  $H^\ell \in \{(K_{1,3})^b, (4P_1)^{b_2}, (4P_1)^{b_3}\}$ . Then we are done by Lemma 6.

Consider a 1-subdivided wall  $G'$  obtained from a wall  $G$ . Recall that 1-subdivided walls are bipartite. Moreover, the vertices that were introduced when subdividing every edge of  $G$  all have degree 2 and form one class of a bipartition  $(B, W)$  of  $G'$ . Let this class be  $B$ . Then  $(K_{1,3})^b$  is not a labelled induced subgraph of  $(B, W, E_{G'})$ . Hence,  $G'$  is weakly  $(K_{1,3})^b$ -free. This means that the class of weakly  $(K_{1,3})^b$ -free graphs contains the class of 1-subdivided walls. As such, it has unbounded clique-width by Lemma 2. The bipartite complement  $G''$  of  $G'$  with respect to  $(B, W)$  is weakly  $(4P_1)^{b_1}$ -free, as  $(K_{1,3})^b$  is the bipartite complement of  $(4P_1)^{b_1}$  and  $(K_{1,3})^b$  is not a labelled induced subgraph of  $(B, W, E_{G'})$ . Hence, the class of weakly  $(4P_1)^{b_1}$ -free graphs has unbounded clique-width by Fact 2. The class of weakly  $(4P_1)^{b_2}$ -free bipartite graphs has unbounded clique-width by Lemma 1 and Theorem 1.  $\square$

### 4.3 The Proof of Theorem 2

*Proof.* We first deal with the bounded cases. First suppose  $H = sP_1$  for some  $s \geq 1$ . Then every  $H$ -free bipartite graph  $G$  has at most  $s - 1$  vertices in each partition class for every bipartition. This means that the clique-width of  $G$  is at most  $2s - 2$ . Now suppose that  $H \in \{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$ . Then the claim follows from combining Lemma 1 with Theorem 1.

We now deal with the unbounded cases. Suppose  $H \neq sP_1$  for some  $s \geq 1$  and that  $H$  is not an induced subgraph of one of the graphs in  $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$ . Then by Lemma 9, either  $H \notin \mathcal{S}$  or,  $H$  is not  $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$ -free. Hence, the clique-width of the class of  $H$ -free bipartite graphs is unbounded by Lemmas 3 and 8, respectively.  $\square$

### 4.4 The Proof of Theorem 3

*Proof.* We first consider the bounded cases. First suppose  $H^\ell = (sP_1)^b$  for some  $s \geq 1$  (the  $H^\ell = (sP_1)^b$  case is equivalent). Then every weakly  $H^\ell$ -free bipartite graph has a bipartition  $(B, W)$  with  $|B| \leq s - 1$ . Hence, the clique-width of such a graph is at most  $s + 1$  (first introduce the vertices of  $B$  by using distinct labels, then use two more labels for the vertices of  $W$ , introducing them one-by-one).

Before considering the case  $H^\ell = (P_1 + P_5)^b$ , we first consider the case where  $H \subseteq_i P_2 + P_4$  or  $H \subseteq_i P_6$ . We first assume that  $H = P_2 + P_4$  or  $H = P_6$ . Then

$H \subseteq_i S_{1,2,3}$ , which implies that the class of  $H$ -free bipartite graphs has bounded clique-width by Theorem 2. All black-and-white labellings of  $P_2 + P_4$  are isomorphic. Similarly, all black-and-white labellings of  $P_6$  are isomorphic. Hence, the class of  $H$ -free bipartite graphs coincides with the class of weakly  $H^\ell$ -free graphs by Lemma 5. We therefore conclude that the latter class also has bounded clique-width.

Now let  $H \subseteq_i P_2 + P_4$  or  $H \subseteq_i P_6$ , but  $H \not\subseteq \{P_2 + P_4, P_6\}$ . Note that  $P_2 + P_4$  and  $P_6$  have a unique labelling  $b$  (up to isomorphism). If  $H^\ell$  is not a labelled induced subgraph of one of  $\{(P_2 + P_4)^b, P_6^b\}$  then  $H$  must have two non-equivalent black-and-white labellings. Since  $H$  is a linear forest, it must have at least two components with an odd number of vertices. Therefore  $H \in \{2P_1, 3P_1, P_1 + P_3, 2P_1 + P_2\}$ . However, in all these cases, for every labelling  $\ell$  of  $H$ ,  $H^\ell \subseteq_{li} P_6^b$  or  $H^\ell \subseteq_{li} (P_2 + P_4)^b$ . Therefore, if  $H \subseteq_i P_2 + P_4$  or  $H \subseteq_i P_6$  then for every labelling  $\ell$  of  $H$ , the weakly  $H^\ell$ -free bipartite graphs are a subclass of either the  $P_6$ -free or  $(P_2 + P_4)$ -free bipartite graphs. In particular, this holds for  $H^\ell = (P_1 + 2P_2)^b$  (we need this observation for the following case).

Finally, suppose  $H^\ell = (P_1 + P_5)^b$ . Let  $G$  be a weakly  $H^\ell$ -free bipartite graph. Then  $G$  has a labelling  $\ell^*$  such that  $H^\ell$  is not a labelled induced subgraph of  $(B_G^{\ell^*}, W_G^{\ell^*}, E_G)$ . If  $|B_G^{\ell^*}|$  is even, then we delete a vertex of  $B_G^{\ell^*}$ . We may do this by Fact 1. Hence  $|B_G^{\ell^*}|$  may be assumed to be odd. Let  $X$  be the subset of  $W_G^{\ell^*}$  that consists of all vertices that are adjacent to less than half of the vertices of  $B_G^{\ell^*}$ . We apply the bipartite complementation between  $X$  and  $B_G^{\ell^*}$ . We may do this by Fact 2. Let  $G_1$  be the resulting bipartite graph, with bipartition classes  $B_{G_1}^{\ell^*} = B_G^{\ell^*}$  and  $W_{G_1}^{\ell^*} = W_G^{\ell^*}$ .

Suppose  $B_{G_1}^{\ell^*}$  contains three vertices  $b_1, b_2, b_3$  and  $W_{G_1}^{\ell^*}$  contains two vertices  $w_1, w_2$  such that  $G_1^{\ell^*}[b_1, b_2, b_3, w_1, w_2]$  is isomorphic to  $(P_1 + 2P_2)^b$ . By construction and because  $|B_{G_1}^{\ell^*}| = |B_G^{\ell^*}|$  is odd,  $w_1$  and  $w_2$  have at least one common neighbour  $b_4 \in B_{G_1}^{\ell^*}$ . Then  $G_1^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$  is isomorphic to  $(P_1 + P_5)^b$ . However, then  $G_1^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$  is also isomorphic to  $(P_1 + P_5)^b$  (irrespective of whether  $w_1$  or  $w_2$  belong to  $X$ ), which is a contradiction. We conclude that  $G_1$  is weakly  $(P_1 + 2P_2)^b$ -free. As observed above, this means that  $G_1$  has bounded clique-width. Hence  $G$  has bounded clique-width.

We now consider the unbounded cases. Let  $H^\ell$  be a labelled bipartite graph that is not isomorphic to one of the (bounded) cases considered already. Suppose that  $H$  contains a cycle or an induced subgraph isomorphic to  $2P_3$ . Then the class of weakly  $H^\ell$ -free graphs has unbounded clique-width by combining Lemma 1 with Theorem 2. Suppose that  $H$  contains a vertex of degree at least 3. Then the class of weakly  $H^\ell$ -free bipartite graphs has unbounded clique-width by Lemma 10(i). It remains to consider the case when  $H = sP_1 + tP_2 + P_r$  for some constants  $1 \leq r \leq 6$ ,  $s \geq 0$  and  $t \geq 0$ , where  $\max\{s, t\} \geq 1$  (as  $H$  is not an induced subgraph of  $P_6$ ).

Suppose  $5 \leq r \leq 6$ . Assume without loss of generality that three vertices of the copy of  $P_r$  in  $H^\ell$  are coloured black. If  $r = 6$  or  $t \geq 1$  or some copy  $P_1$  in  $H^\ell$  is coloured white, or two copies of  $P_1$  in  $H^\ell$  are coloured black, then we can apply Lemma 10(ii). Hence,  $H^\ell = (P_1 + P_5)^b$ , which is not possible by assumption.

Suppose  $r = 4$ . If two vertices in the induced subgraph of  $H^\ell$  isomorphic to  $sP_1 + tP_2$  have the same colour then we can apply Lemma 10(ii). Hence we may assume that  $s \leq 2$  and  $t \leq 1$ , and moreover that  $s = 0$  if  $t = 1$ . Also we would have  $H \subseteq_i P_2 + P_4$  if  $s = 0$  and  $t = 1$  or if  $s = 1$  and  $t = 0$ . Hence, it remains to consider the case  $s = 2$  and  $t = 0$ , such that one copy of  $P_1$  is coloured black and the other one white. In that case, we may apply Lemma 10(ii).

Suppose  $r = 3$ . Assume without loss of generality that the two vertices of the copy of  $P_3$  in  $H^\ell$  are coloured black. Recall that  $s \geq 1$  or  $t \geq 1$ . If  $t \geq 2$ , then we can apply Lemma 10(ii). Suppose  $t = 1$ . Then  $s = 0$  otherwise  $H^\ell$  would contain an induced  $4P_1$  in which not all the vertices are the same colour, in which case we could apply Lemma 10(ii). However, this means that  $H$  is an induced subgraph of  $P_2 + P_4$ . Now suppose  $t = 0$ . Then  $s \geq 2$ , as otherwise  $H$  is an induced subgraph of  $P_2 + P_4$ . If  $s \geq 3$  then  $H^\ell$  contains an induced  $4P_1$  in which not all the vertices are the same colour, in which case we apply Lemma 10(ii). Hence,  $s = 2$  and both copies are coloured black (otherwise we apply Lemma 10(ii)). However, in this case  $H^\ell$  is a labelled induced subgraph of  $(P_1 + P_5)^b$ , which is not possible by assumption.

Finally suppose that  $r \leq 2$ . Then we may write  $H = sP_1 + tP_2$  instead. We must have  $s + t \geq 4$  or  $t \geq 3$ , otherwise  $H$  would be an induced subgraph of  $P_2 + P_4$  or  $P_6$ . If  $t = 0$  then since  $H^\ell \neq (sP_1)^b$  and  $H^\ell \neq (sP_1)^{\bar{b}}$  we can find four copies of  $P_1$  in  $H$  that are not all of the same colour and apply Lemma 10(ii). If  $t \geq 1, s + t \geq 4$ , we can also find four copies of  $P_1$  that are not all of the same colour and apply Lemma 10(ii). Finally, suppose  $s = 0, t = 3$ . In this case we combine Lemmas 1 and 8. This completes the proof.  $\square$

## 5 Conclusions

We have completely determined those bipartite graphs  $H$  for which the class of  $H$ -free bipartite graphs has bounded clique-width. We also characterized exactly those labelled bipartite graphs  $H$  for which the class of weakly  $H$ -free bipartite graphs has bounded clique-width. These results complement the known characterization of Lozin and Volz [18] for strongly  $H$ -free bipartite graphs. A natural direction for further research would be to characterize, for each of the three notions of  $H$ -freeness, the clique-width of classes of  $\mathcal{H}$ -free bipartite graphs when  $\mathcal{H}$  is a set containing at least 2 graphs. In a follow-up paper [12], we apply our results for  $H$ -free bipartite graphs to determine classes of  $(H_1, H_2)$ -free (general) graphs of bounded and unbounded clique-width.

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